Lecture 4 — Limits and Colimits, Diagram Categories, Reedy Model Structures, Properness

Limits and Colimits in Model Categories

Remark 1 (Limits and colimits do not preserve weak equivalences)**.** *Consider the following diagrams in* Top*:*

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & * \\ * & \end{tikzcd} \qquad \begin{tikzcd} S^0 \arrow[r] \arrow[d, hook] & * \\ I & \end{tikzcd}$$

Their pushouts are, respectively, a point $*$ *and* S^1*, with* $* \not\simeq S^1$*.*

Dually, the pullback of $* \to X \leftarrow *$ *is* $*$*, while the pullback of* $* \to X \leftarrow PX$*, where* $* \xrightarrow{W\cap\mathrm{Cof}} PX \xrightarrow{\mathrm{Fib}} X$*, is* $\Omega X \neq *$ *in general.*

In general given a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\sim"'] & B \arrow[r] \arrow[d, "\sim"] & C \arrow[d, "\sim"] \\ A' \arrow[r] & B' \arrow[r] & C' \end{tikzcd}$$

with all vertical arrows weak equivalences, it does ***not*** *follow in that* $A \sqcup_B C \xrightarrow{\sim} A' \sqcup_{B'} C'$*.*

Definition 2. Let D be a small category, C a category. The **diagram category** C^D is the functor category $\mathrm{Fun}(D, C)$. There is a constant-diagram functor:

$$\mathrm{Const}\colon C \longrightarrow C^D, \qquad X \longmapsto \big(\mathrm{Const}\, X\colon i \mapsto X\big).$$

When they exist, the right and left adjoints of Const are the **limit** and **colimit** functors:

$$\begin{tikzcd} C \arrow[r, "\mathrm{Const}"] & C^D \arrow[l, bend right, "\mathrm{colim}"'] \arrow[l, bend left, "\lim"] \end{tikzcd}$$

giving natural isomorphisms

$$C^D(F, \mathrm{Const}\, X) \cong C(\mathrm{colim}\, F, X), \qquad C^D(\mathrm{Const}\, X, F) \cong C(X, \lim F).$$

Definition 3. Let D be a small category, C a model category. Suppose C^D is a model category with W_{C^D} the *objectwise* weak equivalences, and that $\mathrm{Const}\colon C \to C^D$ is a left (resp. right) Quillen functor. Then the **total right (resp. left) derived functors** are:

$$\mathrm{HoLim} := R\lim, \qquad \mathrm{HoColim} := L\,\mathrm{colim}\,.$$

Problem 4. Does a model structure on C^D with objectwise weak equivalences always exist?

Projective and Injective Model Structures

Definition 5 (Projective and Injective model structure)**.** Let D be a small category, C a model category. On the diagram category C^D, in the projective model structure

- a map $f\colon X \to Y$ is a **fibration** if $f_i\colon X_i \to Y_i$ is a fibration in C for all $i \in D$;
- f is a **weak equivalence** if f_i is a weak equivalence for all i;
- f is a **cofibration** if it has the LLP with respect to $W \cap \mathrm{Fib}$.

Dually, on C^D in the injective model structure we have:

- **weak equivalences**: objectwise;
- **cofibrations**: objectwise cofibrations;
- **fibrations**: maps with the RLP with respect to $W \cap \mathrm{Cof}$.

These do **not** exist in general.

Proposition. *In the projective model structure (when it exists):*

- *Trivial cofibrations* $= W \cap \mathrm{Cof}$.
- *Anodyne maps* $= \mathrm{LLP}(\mathrm{Fib})$.

These two classes can differ.

Reedy Model Structures

Theorem 6. *If R is a Reedy category and C is a model category, then C^R admits a canonical model structure (the* Reedy model structure*).*

Definition 7. A **well-ordered set** is a totally ordered set S such that every nonempty subset has a least element. An **ordinal number** α is an isomorphism class of well-ordered sets.

Definition 8. A **Reedy category** is a category R equipped with:

- wide subcategories R_+ and R_-, and
- a **degree function** $d\colon \mathrm{Ob}(R) \to \lambda$ (for some ordinal λ),

such that:

- every nonidentity morphism in R_+ raises degree;
- every nonidentity morphism in R_- lowers degree;
- every morphism f in R factors *uniquely* as $f = f_+ \circ f_-$ with $f_- \in R_-$, $f_+ \in R_+$.

Example 9.

1. Δ is a Reedy category with $d([k]) = k$, $\Delta_+ = \{\text{injective maps}\}$, $\Delta_- = \{\text{surjective maps}\}$.
2. If R is Reedy, so is R^{op} with the same d, $R^{\mathrm{op}}_+ = (R_-)^{\mathrm{op}}$, $R^{\mathrm{op}}_- = (R_+)^{\mathrm{op}}$. In particular Δ^{op} is Reedy.
3. Every ordinal α is a Reedy category (viewed as a poset): $a \to b \Leftrightarrow a < b$, with $\alpha_+ = \alpha$, $\alpha_- = \{\text{identities}\}$. In particular $0 \to 1 \to 2 \to \cdots$ is Reedy. Both $\cdots \to \bullet \to \bullet$ and $\bullet \to \bullet \to \cdots$ are Reedy.
4. The categories $\bullet \leftarrow \bullet \to \bullet$, $\bullet \rightrightarrows \bullet$ are Reedy.

Definition 10. Let R be a Reedy category, C a model category, $X \in C^R$ a diagram, and $n \in R$.

- The **latching object** of X at n is
$$L_r X := \operatorname{colim}_{s \xrightarrow{+} r} X_s.$$
(colimit over the full subcategory R_+/r minus the identity at r.)
- The **matching object** of X at r is
$$M_r X := \lim_{r \xrightarrow{-} s} X_s.$$
(limit over r/R_- minus the identity at r.)

There are always natural maps $L_r X \to X_r \to M_r X$.

Definition 11 (Reedy model structure)**.** Let R be a Reedy category, C a model category. A map $f \colon X \to Y$ in C^R is:

- a **Reedy equivalence** if $f_r \in W_C$ for all $r \in R$;
- a **Reedy cofibration** if for all $r \in R$, the induced map
$$L_r Y \coprod_{L_r X} X_r \longrightarrow Y_r$$
is a cofibration in C (replace cofibration by trivial cofibration for Reedy trivial cofibrations);
- a **Reedy fibration** if for all $r \in R$, the induced map
$$X_r \longrightarrow Y_r \times_{M_r Y} M_r X$$
is a fibration in C (replace by trivial fibration for Reedy trivial fibrations).

Remark. $X \in C^R$ *is* ***Reedy cofibrant*** *iff* $L_r X \to X_r \in \mathrm{Cof}$ *for all* r*.* X *is* ***Reedy fibrant*** *iff* $X_r \to M_r X \in \mathrm{Fib}$ *for all* r*.*

Properness

Definition. A model category C is **left proper** if pushouts of weak equivalences along cofibrations are weak equivalences: whenever

$$\begin{tikzcd} A \arrow[r, "c \in \mathrm{Cof}"] \arrow[d, "w \in W"'] & B \arrow[d, "v"] \\ C \arrow[r] & D \end{tikzcd}$$

is a pushout, then $v \in W$.

C is **right proper** if pullbacks of weak equivalences along fibrations are weak equivalences: whenever

$$\begin{tikzcd} X \arrow[r] \arrow[d, "t"'] & Y \arrow[d, "p \in \mathrm{Fib}"] \\ Z \arrow[r, "u \in W"'] & V \end{tikzcd}$$

is a pullback, then $t \in W$.

Proposition 12.

- *If $C_c = C$ (every object is cofibrant), then C is left proper.*
- *If $C_f = C$ (every object is fibrant), then C is right proper.*

Remark (Properness via slice categories)**.** *Properness concerns only the behavior of weak equivalences. Given $f\colon X \to Y$ in a model category C, the slice categories C/X and C/Y are model categories, and postcomposition f_* and pullback f^* give a Quillen adjunction $f_* \dashv f^*$.*

Theorem: *C is right proper if and only if $f \in W \Rightarrow f_* \dashv f^*$ is a Quillen equivalence.*

Homotopy Pushouts and Pullbacks

Example (Homotopy pushouts)**.** We have seen that, given

$$\begin{tikzcd} A \arrow[d, "\sim"'] & B \arrow[l, "a'"'] \arrow[r, "c'"] \arrow[d, "\sim"'] & C \arrow[d, "\sim"'] \\ A' & B' \arrow[l] \arrow[r] & C', \end{tikzcd}$$

with all vertical maps weak equivalences, the induced map between pushouts need not be a weak equivalence.

The **homotopy pushout** of $A \leftarrow B \to C$ is defined as the standard pushout of a *cofibrant replacement*:

$$\begin{tikzcd} \widetilde{A} \arrow[d, "\sim"'] & \widetilde{B} \arrow[l, "a'"'] \arrow[r, "c'"] \arrow[d, "\sim"'] & \widetilde{C} \arrow[d, "\sim"'] \\ A & B \arrow[l] \arrow[r] & C, \end{tikzcd}$$

where $a', c' \in \mathrm{Cof}$ and $\widetilde{A}, \widetilde{B}, \widetilde{C} \in C_c$. The diagram $\widetilde{A} \leftarrow \widetilde{B} \to \widetilde{C}$ is a **cofibrant diagram**.

We need even less. A square

$$\begin{tikzcd}
B \arrow[r, "c"] \arrow[d, "a"'] & C \arrow[d] \\
A \arrow[r] & A \cup_B C
\end{tikzcd}$$

is a **homotopy pushout** if either of the following holds:

1. $C, B, A \in C_c$ and $a \in \mathrm{Cof}$;

2. $a \in \mathrm{Cof}$ and C is left proper.

Example (Homotopy pullbacks)**.** A diagram $X \overset{x}{\to} Y \overset{z}{\leftarrow} Z$ is **fibrant** if $x, z \in \mathrm{Fib}$ and $X, Y, Z \in C_f$.

Given a commutative diagram

$$\begin{tikzcd}
Q \arrow[r] \arrow[d, "\sim"'] & R \arrow[d, "\sim"] & S \arrow[l] \arrow[d, "\sim"] \\
X \arrow[r] & Y & Z, \arrow[l]
\end{tikzcd}$$

the standard pullback $X \times_Y Z$ is the **homotopy pullback** of $Q \to R \leftarrow S$.

Vice versa, the pullback is a homotopy pullback if either:

1. $X, Y, Z \in C_f$ and $x \in \mathrm{Fib}$; or

2. $x \in \mathrm{Fib}$ and C is right proper.